

Characteristic Numbers and Generalized Path Integrals

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To Raoul Bott in honor of his 70th birthday

I hope in this exposition to touch on two themes which are among Professor Bott's many mathematical interests. The first—characteristic numbers—appears throughout his work. My second topic, path integrals, have been around in physics since they were introduced by Feynman in the late 1940s, but have only recently been applied to problems of purely geometric interest. Edward Witten has led the way in this “topological quantum field theory”, which has attracted the enthusiasm of many mathematicians. The ideas of quantum field theory, though far from mathematically understood, seem to provide a unified framework for many recent invariants in low dimensional topology (à la Donaldson, Jones, Casson, Floer, . . .) and introduce some new invariants as well. Our first goal is to explain that ideas of (classical) field theory inspire new insight into characteristic numbers. The second goal is to explain how summing over fields—the path integral—produces diffeomorphism invariants which satisfy *gluing laws*, and how a generalization of this idea explains some of the algebraic structure behind these invariants in three dimensions. We can only do this rigorously in a “toy model”, where the path integral reduces to a finite sum, but we hope that the ideas here will shed light on more interesting examples as well. The toy model is based on a characteristic number for finite principal covering spaces. As I have explained most of these ideas elsewhere, my goal here is to give an elementary account of the basic concepts and the simplest cases.

The oldest, and in some sense most intriguing, characteristic number is the Euler number of a manifold. More generally, an oriented real vector bundle of rank n over a closed oriented manifold of dimension n has an Euler number, which takes values in the integers. Real and complex vector bundles over closed oriented manifolds have other characteristic numbers due to Chern, Pontrjagin,

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and Stiefel-Whitney. These integers are *primary* characteristic numbers in *cohomology*; they are topological invariants. There are similar *primary* characteristic numbers in *K-theory* for real and complex bundles over compact oriented (or spin) manifolds. The topological definition [AH] of these characteristic numbers uses Bott Periodicity in an essential way. They include the signature of a manifold, the \hat{A} -genus, etc. What I term *secondary characteristic numbers* are *geometric* invariants which depend on a metric or connection and take values in \mathbb{R}/\mathbb{Z} , or, after exponentiating, in the circle group \mathbb{T} . In cohomology they are the Chern-Simons numbers and in *K-theory* they are the η -invariants. We summarize in Table 1. Our observation is that these classical invariants extend to invariants of manifolds “below the top dimension” and of manifolds with boundary. Thus, for example, the characteristic number c_4 is an integer associated to a complex vector bundle over a closed oriented 8-manifold. There are invariants of complex vector bundles over compact oriented manifolds (possibly with boundary) in all dimensions ≤ 8 , but now the invariants are not simply numbers. For manifolds of dimension 7 they are sets, for manifolds of dimension 6 they are categories, and so on. These sets and categories have more structure, of course. These extended invariants obey gluing laws, so are in some sense *local* invariants. That is, in some respect they behave in some respects like the integral over the manifold of a locally computed density. In §1 we explain the simplest case of these ideas: the Euler number of a complex line bundle over a surface. The structure of the invariants in other cases is similar, though the actual constructions are quite different. I do not know all of the constructions for the *K-theory* invariants.

	cohomology	<i>K</i>-theory
primary	Chern numbers, Pontrjagin numbers	signature, \hat{A} -genus
secondary	Chern-Simons invariants	η -invariants

Table 1: Classical characteristic numbers

Because the invariants in Table 1 obey gluing laws, they are appropriate classical actions for a field theory. Whereas the classical action is an “integral” over a finite dimensional manifold, the quantum path integral is an “integral” over a space of fields on that manifold, which typically is infinite dimensional. Although mathematical physicists have made great progress in understanding the technology of these integrals, the cases of topological significance discussed here remain elusive. Our main idea in §2 is to extend this (ill-defined) notion of integration over fields to the *extended* classical action, i.e., to fields on manifolds below the top dimension. For example, in a three dimensional topological field theory the path integral for a closed oriented 3-manifold produces a complex number, whereas canonical quantization assigns a complex Hilbert space to a closed oriented 2-manifold. These are usually considered as different processes. We reinterpret this Hilbert

space as the result of an integration process over the space of fields on the 2-manifold. The extended classical action has values which *are* hermitian lines, and the integral (direct sum) of hermitian lines is a Hilbert space. This is very strange—integrals are usually numbers, not Hilbert spaces! Nonetheless, this strange notion of integration leads immediately to a gluing law for the Hilbert spaces which, for example, is closely related to the “Verlinde formula” in conformal field theory. This idea is a pure formality in general; we emphasize that so far this whole scheme only works rigorously for our toy model. After all, the usual path integral is also not rigorously defined in topological theories. Our extended notion of path integral applies to 1-manifolds (in a three dimensional topological theory) and leads to a braided monoidal category, and so ultimately to a quantum group. We view this as the *solution* to the theory. That is, we start with the classical action to define the theory, then compute the quantum group (actually its category of representations) from the theory, and finally derive formulas for invariants from the gluing laws of the path integral (e.g. in terms of a presentation as surgery on a link).

	cohomology	<i>K</i> -theory
primary	Donaldson invariants, Floer homology, Gromov invariants	???
secondary	Reshetikhin-Turaev-Witten invariants, Jones invariants	???

Table 2: Quantum characteristic numbers

I find it useful to organize many of the topological quantum field theories floating around into Table 2. I like to think of the invariants in these theories as *quantum characteristic numbers*. Often people distinguish between two types of topological field theories, which here is the distinction between primary and secondary invariants. Notice that on the quantum level the geometric data used to define the secondary characteristic number has been integrated out, so we are left with topological invariants in both types of theory. The main example here is Witten’s definition [W1] of an invariant of 3-manifolds as a path integral of Chern-Simons. This is most closely analogous to the usual path integral in physics. Less clear is the integration process which leads from primary characteristic numbers (which are already topological invariants) to Donaldson invariants, etc. Witten introduced *supersymmetric* path integrals [W2], [W3] to explain these invariants, and Baulieu-Singer reinterpreted this in terms of primary characteristic numbers and gauge fixing [BS]. We should also mention that these invariants are in some sense an infinite dimensional Euler number, and that the path integral representation is an infinite dimensional version of the Mathai-Quillen formula [MQ], [AJ]. From this point of view the primary characteristic numbers in

K -theory which I placed in Table 1 could be pigeon-holed in the upper left hand corner of Table 2, thanks to supersymmetric quantum mechanics. Just as with the classical invariants in Table 1, the geometric and topological applications of *primary* quantum invariants (Donaldson) have been more striking than the applications of *secondary* invariants (Chern-Simons).

The idea that in a three dimensional topological quantum field theory one should attach certain types of categories to 1-manifolds has been discussed by Kazhdan, Segal, Lawrence, Yetter, Crane, and many others. The construction by generalized path integrals is new. I refer the reader to [F1] for more details about generalized path integrals and for the detailed computations for finite group gauge theory. The appendix to [FQ] defines the characteristic numbers used in this model, and [FQ] contains many more details about the basics of the theory. There is another expository account of some of this material in [F2]. In particular, the derivation of the quantum group is explained in more detail there.

It is a great pleasure and honor to dedicate this paper to Raoul Bott. His birthday is a wonderful occasion to celebrate his youthful exuberance for life and for mathematics. *L'chaim*, Raoul!

§1 CHARACTERISTIC NUMBERS

Consider a complex line bundle $L \rightarrow X^2$ over a closed oriented 2-manifold X . Its Euler number $e_X(L) \in \mathbb{Z}$ can be computed in several ways. For example, choose any section $s: X \rightarrow L$ which is transverse to the zero section. Then s has isolated zeros, and the orientations determine a sign $\text{sgn}(y)$ for each zero y . A standard argument shows that

$$(1.1) \quad e_X(L) = \sum_{y \in \text{Zero}(s)} \text{sgn}(y)$$

is independent of s , and so defines a topological invariant $e_X(L) \in \mathbb{Z}$. As we deform s two zeros of opposite sign can simultaneously die or simultaneously appear. But the total signed number of zeros is constant.

Consider now the same situation over a compact oriented 2-manifold X with nonempty boundary. Assume that the section s does not vanish on ∂X . It is still true that (1.1) does not change if we modify s in the interior of X , but now (1.1) depends on the restriction of s to the boundary. If s is deformed allowing zeros on the boundary, then (1.1) changes according to the number of zeros (counted with sign, of course) which flow through the boundary during the deformation. More formally, define $e_X(L, t)$ for any nonvanishing section $t: \partial X \rightarrow L$ by extending to a section $s: X \rightarrow L$ and counting zeros as in (1.1). Then for two such section t, t' we have

$$(1.2) \quad e_X(L, t') - e_X(L, t) = \deg_{\partial X}(t'/t).$$

Here the ratio t'/t is a map $\partial X \rightarrow \mathbb{C}^\times$ which has a degree, or winding number, computed using the induced orientation of ∂X . So the relative Euler number depends on the section t (up to homotopy). This is the traditional point of view on relative characteristic numbers—they depend on a trivialization on the boundary.

Now the new twist: We can define a relative Euler number which is independent of t if we abandon the idea that it should be a number. Namely, let Sect^* denote the set of nonzero sections, and set

$$(1.3) \quad T_{\partial X}(\partial L) = \{e: \text{Sect}^*(\partial L) \longrightarrow \mathbb{Z} \text{ which obey } e(t') - e(t) = \deg_{\partial X}(t'/t)\}.$$

Notice that if $e, e' \in T_{\partial X}(\partial L)$ then $e' - e$ is a constant integer. Likewise, if $e \in T_{\partial X}(\partial L)$ and $n \in \mathbb{Z}$ then $e + n \in T_{\partial X}(\partial L)$. So $T_{\partial X}(\partial L)$ is a principal homogeneous space for the integers, a so-called \mathbb{Z} -torsor. It is an “affine” copy of \mathbb{Z} —a copy of \mathbb{Z} without a preferred origin. In other terms it is a principal \mathbb{Z} bundle over a point. It is *not* a group. The function $e_X(L, \cdot)$ lies in this torsor, by (1.2), and so we obtain a relative Euler number

$$(1.4) \quad e_X(L) \in T_{\partial X}(\partial L)$$

which only depends on the bundle L .

We now explain the sense in which (1.4) is a topological invariant. First, notice that (1.3) defines a \mathbb{Z} -torsor $T_Y(K)$ for any line bundle $K \rightarrow Y$ over any closed oriented 1-manifold Y . It is a topological invariant of K . Usually topological invariants are *numbers*, and equivalent objects have *equal* invariants. Here the topological invariant is a *set*, and equivalent objects have *isomorphic* invariants. Precisely, if $K' \xrightarrow{\psi} K$ is an isomorphism of line bundles over Y , it induces an isomorphism of \mathbb{Z} -torsors $T_Y(K') \xrightarrow{\psi_*} T_Y(K)$. This is familiar from algebraic topology, where invariants like the fundamental group or homology groups of a space are sets and homeomorphisms on spaces induce isomorphisms of these sets. The relative Euler number (1.4) is a topological invariant in this sense: If $L' \xrightarrow{\varphi} L$ is an isomorphism of circle bundles over X , then $(\partial\varphi_*)(e(L')) = e(L)$. Most importantly, the Euler number obeys a *gluing law*. To see this, first note that if $-Y$ denotes Y with the opposite orientation, then for any $K \rightarrow Y$ there is a natural pairing

$$(1.5) \quad +: T_Y(K) \times T_{-Y}(K) \longrightarrow \mathbb{Z}$$

by addition. Now suppose X_1, X_2 are oriented surfaces with boundaries $\partial X_1 = Y$ and $\partial X_2 = -Y$. Let $X = X_1 \cup_Y X_2$ denote the surface formed by gluing along the boundary. Suppose we are also given circle bundles $L_i \rightarrow X_i$ and an isomorphism $\partial L_1 \rightarrow \partial L_2$. Let $L \rightarrow X$ denote the glued bundle. Then the gluing law asserts

$$(1.6) \quad e_X(L) = e_{X_1}(L_1) + e_{X_2}(L_2).$$

The proof is direct from the definitions.

There are alternative constructions of these invariants. For example, fix a metric on L and let \mathcal{A}_L denote the space of unitary connections on L . Let $F(\theta) \in \Omega_X^2$ denote $i/2\pi$ times the curvature of a connection $\theta \in \mathcal{A}_L$. Then

$$(1.7) \quad e_X(L, \theta) = \int_X F(\theta)$$

is a real-valued function on \mathcal{A}_L . If $\partial X = \emptyset$ it is a constant integer equal to $e_X(L)$. If $\partial X \neq \emptyset$ then

$$\exp\left(2\pi i \int_X F(\theta)\right) = \text{hol}_{\partial X}(\theta),$$

where $\text{hol}_{\partial X}(\theta)$ is the holonomy of θ around ∂X . In this situation we define the \mathbb{Z} -torsor

$$(1.8) \quad T'_{\partial X}(\partial L) = \{f: \mathcal{A}_{\partial L} \rightarrow \mathbb{R} : e^{2\pi i f(\theta)} = \text{hol}_{\partial X}(\theta)\}.$$

Then (1.7) defines the relative Euler number. With these definitions the gluing law (1.6) follows from the additivity of integration:

$$\int_X = \int_{X_1} + \int_{X_2}.$$

The Euler number is an example of a primary characteristic number (see Table 1). More generally, if G is a Lie group and $\lambda \in H^n(BG; \mathbb{Z})$ a universal characteristic class, then λ determines¹ a characteristic number $\lambda_X(P)$ for a G bundle $P \rightarrow X$ over a closed oriented n -manifold X . One can define a \mathbb{Z} -torsor $T_Y(Q)$ for a G bundle $Q \rightarrow Y$ over a closed oriented $(n-1)$ -manifold Y , and a relative invariant for n -manifolds with boundary. The story continues to higher codimensions. For example, the invariant of a bundle over a closed oriented $(n-2)$ -manifold is a certain type of category, called a *gerbe*.

We illustrate the codimension 2 invariant in the simplest case of the Euler number. Recall the definition (1.3) of the \mathbb{Z} -torsor associated to a line bundle $K \rightarrow Y$ over a closed oriented surface. Now suppose Y is a compact oriented 1-manifold with boundary, i.e., a finite union of circles and closed intervals. Let $u: \partial Y \rightarrow \partial K$ be a nonzero section over the boundary and let $\text{Sect}^*(K, u)$ be the set of nonzero sections $t: Y \rightarrow K$ with $\partial t = u$. Then define the \mathbb{Z} -torsor

$$T_Y(K, u) = \{e: \text{Sect}^*(K, u) \rightarrow \mathbb{Z} : e(t') - e(t) = \deg_{\partial X}(t'/t)\}.$$

The degree is well-defined since $\partial t' = \partial t$. We need to determine the dependence of $T_Y(K, u)$ on u . Now if u' is any other trivialization of $\partial K \rightarrow \partial Y$, then the set of nonzero paths joining u to u' , up to homotopy, is a \mathbb{Z} -torsor $T_{\partial Y}(\partial K, u, u')$. Such a path gives a 1:1 correspondence $\text{Sect}^*(K, u) \rightarrow \text{Sect}^*(K, u')$, and so there is an isomorphism

$$T_Y(K, u) \times T_{\partial Y}(\partial K, u, u') \longrightarrow T_Y(K, u').$$

This leads us to define

$$(1.9) \quad \mathcal{G}_{\partial Y}(\partial K) = \{T: \text{Sect}^*(\partial K) \rightarrow \{\mathbb{Z}\text{-torsors}\} \text{ with given isomorphisms} \\ T(u) \times T_{\partial Y}(\partial K, u, u') \rightarrow T(u') \text{ which satisfy various conditions}\}.$$

In this definition T is a set-valued function, so $\mathcal{G}_{\partial Y}(\partial K)$ is a collection of set-valued functions, in particular, a category. The “various conditions” are related to the category structure and describe an associativity constraint for the isomorphism associated to three nonzero sections u, u', u'' . Think

¹In fact, we must choose a particular representative of the cohomology class $\lambda \in H^n(BG; \mathbb{Z})$ to carry out the constructions which follow [FQ, Appendix B], [F1, §2].

of the *gerbe* (1.9) as an affine copy of the category of \mathbb{Z} -torsors. The relative invariant $T_Y(K)$ lies in the gerbe $\mathcal{G}_{\partial Y}(\partial K)$ by definition.

This invariant admittedly contains little information. However, if we start with a 4 dimensional characteristic class, then the gerbe attached to a surface leads to a central extension of the diffeomorphism group of the surface which arises in quantum Chern-Simons theory [W1]. (It is better here to start with the *signature* of a 4-manifold [F2].) The central extension is often realized in terms of “2-framings”.

Our interest in §2 is in 3 dimensional invariants of principal bundles with *finite* gauge group G . But if G is finite then $H^n(BG; \mathbb{Z})$ consists entirely of torsion elements for $n > 0$, and so the primary integral characteristic numbers vanish. Rather, we start with

$$(1.10) \quad \lambda \in H^3(BG; \mathbb{R}/\mathbb{Z}) \cong H^4(BG; \mathbb{Z})$$

and obtain a characteristic number $\lambda_X(P) \in \mathbb{R}/\mathbb{Z}$ for G bundles $P \rightarrow X^3$ over a closed oriented 3-manifold. Exponentiating, the invariant

$$(1.11) \quad e^{2\pi i \lambda_X(P)} \in \mathbb{T}$$

lies in the circle group \mathbb{T} of unit norm complex numbers. The relative picture is similar: To a G bundle $Q \rightarrow Y$ over a closed oriented surface Y we attach a \mathbb{T} -torsor, and to a bundle $P \rightarrow X$ over a 3-manifold with boundary the relative invariant lives in the \mathbb{T} -torsor of the boundary. These invariants obey a gluing law (1.6) which we now write multiplicatively. Notice that any \mathbb{T} -torsor is the set of unit norm elements in a hermitian line, which here we denote $L_Y(Q)$. So the relative invariant of $P \rightarrow X$ is

$$(1.12) \quad e^{2\pi i \lambda_X(P)} \in L_{\partial X}(\partial P).$$

Similarly, the invariant of a G bundle $R \rightarrow S$ over a closed oriented 1-manifold S (finite union of circles) is a \mathbb{T} -gerbe $\mathcal{G}(R)$, which we think of as an affine copy of the category of hermitian lines. (Imagine definition (1.9) with hermitian line-valued functions.) Then the relative invariant of a G bundle $Q \rightarrow Y$ over an oriented surface with boundary is an element

$$(1.13) \quad L_Y(Q) \in \mathcal{G}_{\partial Y}(\partial Q).$$

Although (1.11) is a topological invariant, and in that sense is primary (Table 1), because of the transgression in (1.10) we can think of it as a Chern-Simons invariant for bundles with finite structure group. More generally, for any compact group G and class $\lambda \in H^n(BG; \mathbb{Z})$ there are

secondary *geometric* invariants for connections on bundles over compact oriented manifolds of dimension at most $n - 1$. For a closed oriented $(n - 1)$ -manifold the invariant lies in \mathbb{T} , for a closed oriented $(n - 2)$ -manifold the invariant is a \mathbb{T} -torsor (or hermitian line), etc. For a family of connections on an $(n - 2)$ -manifold there is a connection on the line bundle over the parameter space. We develop these ideas in [F3] using a variant of the *Čech-de Rham complex* (more fondly “tic-tac-toe” [BT]) which is related to Deligne cohomology [Bry]. This amounts to a generalization of the theory of differential forms, including both differentiation and integration. The case $n - 1 = 3$ is of special interest [F4] because of the relation with special geometric structures in low dimensional gauge theory and with the corresponding quantum invariants.

Let us briefly consider the K -theory side of Table 1. From the point of view of Dirac operators the primary characteristic number—in \mathbb{Z} —is the index of a Dirac operator on an even dimensional closed spin manifold, and the secondary characteristic number—in \mathbb{T} —is the exponentiated ξ -invariant of a Dirac operator on an odd dimensional closed spin manifold. (Recall that the ξ -invariant is half the η -invariant plus half the dimension of the kernel.) Notice that the former is a topological invariant whereas the latter depends on geometric data, i.e., a metric and possibly a connection. One can define relative invariants in this context. First, a reinterpretation of the work of Atiyah-Patodi-Singer [APS] constructs the index of a Dirac operator on a manifold with boundary as a topological invariant [F5]. It lives in a \mathbb{Z} -torsor (which is an invariant of the Dirac operator on the boundary) and satisfies a gluing law. This is completely analogous to (1.4) and (1.6). In recent work with Xianzhe Dai [DF] we construct a relative exponentiated ξ -invariant (analogous to (1.12)) which lives in the determinant line of the Dirac operator on the boundary and satisfies a gluing law. However, I do not know how to extend these constructions to manifolds of lower dimension (except for the next step on the “primary side”). Also, it would be interesting to give topological constructions of the relative primary invariants which generalize the direct image [AH].

§2 GENERALIZED PATH INTEGRALS

Consider a finite group G and a class $\lambda \in H^3(BG; \mathbb{R}/\mathbb{Z})$. In the last section we indicated that there are invariants $\lambda_X(P) \in \mathbb{T}$ for each principal G bundle $P \rightarrow X^3$ over a closed oriented 3-manifold. If we are interested in constructing topological invariants of X , then we need to eliminate the dependence on P . One way—the physicists’ way—is to integrate over P . In quantum field theory P is called a *field* and this integral over the space of fields is called the *path integral*. It is in this sense that our discussion here is a field theory. More specifically, it is the quantum Chern-Simons theory [W1] for finite gauge group, first introduced by Dijkgraaf and Witten [DW]. Fortunately, the integral reduces to a finite sum in this theory and there are no analytic problems to worry about.

Let \mathcal{C}_X denote the collection of G bundles over X . It is a category—morphisms are bundle maps which cover the identity. Let $\overline{\mathcal{C}_X}$ denote the finite set of equivalence classes of bundles, and define a measure on $\overline{\mathcal{C}_X}$ by

$$(2.1) \quad \mu_X(P) = \frac{1}{\# \text{Aut } P}$$

for $P \in \mathcal{C}_X$. The topological invariant of a closed oriented 3-manifold X we wish to study is

$$(2.2) \quad Z_X = \int_{\overline{\mathcal{C}_X}} e^{2\pi i \lambda_X(P)} d\mu_X(P).$$

We view the integrand (1.11) as a complex number and perform the integral by summing complex numbers. So $Z_X \in \mathbb{C}$. Although we use integral notation, (2.2) is a finite sum. The fact that $e^{2\pi i \lambda_X(P)}$ is a topological invariant of P quickly leads to a proof that Z_X is a topological invariant of X .

Just as with the Euler number in §1, we would like to define an invariant of a 3-manifold with boundary which satisfies a gluing law analogous to (1.6). Let X be a closed oriented 3-manifold and $Y \hookrightarrow X$ an embedded closed oriented surface. Denote by X^{cut} the manifold obtained by cutting X along Y . (See Figure 1.) Notice that $\partial X^{\text{cut}} = Y \sqcup -Y$ is a disjoint union of two copies of Y . The fields (bundles) fit into the following diagram:

$$\begin{array}{ccc} \mathcal{C}_X & \xrightarrow{c} & \mathcal{C}_{X^{\text{cut}}} \\ r_1 \downarrow & & \downarrow r_2 \\ \mathcal{C}_Y & \xrightarrow{\Delta} & \mathcal{C}_Y \times \mathcal{C}_{-Y} \end{array}$$

The vertical arrow r_1 is restriction to Y , the arrow r_2 is restriction to ∂X^{cut} , the arrow Δ is the diagonal inclusion, and c is the pullback under the gluing map. For the moment we ignore

symmetries and pretend that (2.2) is an integral over \mathcal{C}_X . Then we propose to do the integral over \mathcal{C}_X in two stages using Fubini's theorem: First integrate over the fibers of r_1 and then over \mathcal{C}_Y . Now the gluing law (1.9) for λ says that if $P \in \mathcal{C}_X$ and $P^{\text{cut}} = c(P) \in \mathcal{C}_{X^{\text{cut}}}$, then

$$(2.3) \quad e^{2\pi i \lambda_X(P)} = e^{2\pi i \lambda_{X^{\text{cut}}}(P^{\text{cut}})}.$$

The right hand side of (2.3) lives in $L_Y(Q) \otimes L_{-Y}(Q)$ (cf. (1.12)) which is identified with \mathbb{C} via a pairing analogous to (1.5). Our hope, then, is to make the following computation:

$$(2.4) \quad \begin{aligned} Z_X &= \int_{\mathcal{C}_X} e^{2\pi i \lambda_X(P)} d\mu_X(P) = \int_{\mathcal{C}_Y} \int_{r_1^{-1}(Q)} e^{2\pi i \lambda_X(P)} d\mu_{r_1^{-1}(Q)}(P) d\mu_Y(Q) \\ &= \int_{\mathcal{C}_Y} \int_{r_2^{-1}(Q, Q)} e^{2\pi i \lambda_{X^{\text{cut}}}(P^{\text{cut}})} d\mu_{r_2^{-1}(Q, Q)}(P^{\text{cut}}) d\mu_Y(Q). \end{aligned}$$

For this to be a valid computation we need the measures to work out properly. Also, we must include the symmetries. Both are easily handled [FQ, §2].

Let's reinterpret the last line of (2.4). For a moment suppose that X' is any compact oriented 3-manifold with boundary. Then for $Q \in \mathcal{C}_{\partial X'}$ define

$$\mathcal{C}_{X'}(Q) = \{P \rightarrow X' \text{ such that } \partial P = Q\}.$$

There are symmetries here as well, and we are deliberately vague in order to keep the ideas as simple as possible. Note that $\mathcal{C}_{X'}(Q)$ is the space of fields with a given fixed boundary value. We generalize the path integral (2.2) to manifolds with boundary by defining

$$(2.5) \quad Z_{X'}(Q) = \int_{\mathcal{C}_{X'}(Q)} e^{2\pi i \lambda_{X'}(P)} d\mu_{X'}(P).$$

The right hand side takes values in the hermitian line $L_{\partial X'}(Q)$ (cf. (1.12)). So $Z_{X'}$ is a section of the hermitian line bundle $L_{\partial X'} \rightarrow \mathcal{C}_{\partial X'}$. In an appropriate sense it is invariant under symmetries. For any closed surface Y set

$$(2.6) \quad E(Y) = \text{invariant sections of } L_Y \rightarrow \mathcal{C}_Y.$$

Then (2.5) determines a relative invariant

$$Z_{X'} \in E(\partial X').$$

We impose an L^2 inner product on (2.6) using the measure (2.1) on *equivalence classes* of bundles.

With these definitions the gluing law (2.4) (with symmetries restored) takes the form

$$(2.7) \quad \begin{aligned} Z_X &= \int_{\overline{\mathcal{C}_{\partial X}}} Z_{X^{\text{cut}}}(Q, Q) d\mu_{\partial X}(Q) \\ &= \text{Trace}_{\partial X}(Z_{X^{\text{cut}}}), \end{aligned}$$

where $\text{Trace}_{\partial X}: E(\partial X) \otimes E(-\partial X) \rightarrow \mathbb{C}$ is formed using the L^2 inner product. (Notice that $L_{-\partial X}(Q) \cong \overline{L_{\partial X}(Q)}$ and so $E(-\partial X) \cong \overline{E(\partial X)}$.)

What we have recounted so far is the standard argument that path integrals give numerical invariants which satisfy gluing laws. The relative invariants in quantum theories do not live in “one dimensional” torsors, as in the classical case (1.3), (1.8), but rather in Hilbert spaces (2.6). Now we want to go further and derive gluing laws for the Hilbert spaces.

Here is the main idea: Re-express the Hilbert space $E(Y)$ as an integral over the space of fields $\overline{\mathcal{C}_Y}$. Then repeat the argument which leads from (2.2) to the gluing law (2.7). In the process we will define an invariant $\mathcal{E}(S)$ of a closed oriented 1-manifold S , by analogy with (2.6). If the latter can be re-expressed as an integral we can again iterate the process.

To find the integral which computes $E(Y)$ recall that the characteristic class λ associates to each G bundle $Q \rightarrow Y$ a hermitian line $L_Y(Q)$. Furthermore, an isomorphism $Q' \rightarrow Q$ of bundles induces an isomorphism $L_Y(Q') \rightarrow L_Y(Q)$ of hermitian lines. So there is a quotient hermitian line bundle

$$L_Y \longrightarrow \overline{\mathcal{C}_Y}$$

(This bundle degenerates where automorphisms of $Q \in \mathcal{C}_Y$ act nontrivially on $L_Y(Q)$.) Our formula for the Hilbert space is:

$$(2.8) \quad \boxed{E(Y) = \int_{\overline{\mathcal{C}_Y}} L_Y(Q) d\mu_Y(Q)}$$

This is a boldly presented formula and it requires some explanation. The right hand side has the form

$$(2.9) \quad \sum_{\text{finite}} (\text{positive number}) \cdot (\text{hermitian line}).$$

If $\mu > 0$ and L is a hermitian line, define $\mu \cdot L$ to be the hermitian line with the same underlying vector space as L but with an inner product which is μ times the inner product on L . Interpret the finite sum as the direct sum of hermitian lines. This is the sense in which the right hand side of (2.8) defines a Hilbert space—as a sum of hermitian lines—and it is easy to see that

(2.6) and (2.8) define the same Hilbert space. As explained in §1 the hermitian line $L_Y(Q)$ is the two dimensional counterpart of the three dimensional invariant $e^{2\pi i \lambda_X(P)}$. So the integrals (2.2) and (2.8) have the same form: they are the integral of the exponentiated classical action over the space of fields.

By now it should be clear how to continue. If Y is an oriented surface with boundary, then by analogy with (2.5) we define

$$E(Y)(R) = \int_{\mathcal{C}_Y(R)} L_Y(Q) d\mu_Y(Q), \quad R \in \mathcal{C}_{\partial Y}.$$

Recall from (1.13) that $L_Y(Q)$ is an element of the gerbe $\mathcal{G}_{\partial Y}(R)$. Think of the latter as an affine copy of the category of hermitian lines. Then just as we can add hermitian lines to obtain a Hilbert space (2.9), we can add elements of $\mathcal{G}_{\partial Y}(R)$. What do we get? If $\mathcal{G}_{\partial Y}(R)$ were trivial we would get a Hilbert space. Then $E(Y)$ would be an invariant function from $\mathcal{C}_{\partial Y}$ to the category of Hilbert spaces. The nontriviality of $\mathcal{G}_{\partial Y}(R)$ means that $E(Y)$ is an invariant function from $\mathcal{C}_{\partial Y}$ to an affine version of the category of Hilbert spaces, a so-called *2-Hilbert space* [KV]. The space $\mathcal{E}(\partial Y)$ of all such functions is again a 2-Hilbert space and $E(Y) \in \mathcal{E}(\partial Y)$. If we cut a surface along a circle, then we can formulate a gluing law analogous to (2.7). Finally, for any closed oriented 1-manifold S we write the 2-Hilbert space $\mathcal{E}(S)$ as an integral:

$$(2.10) \quad \mathcal{E}(S) = \int_{\mathcal{C}_S} \mathcal{G}_S(R) d\mu_S(R).$$

In principle, we can continue to even lower dimensions.

We stop here, but refer the reader to [F2] for an expository account which begins with the notion of a 2-Hilbert space. There we give a general argument to show how in a 3 dimensional topological quantum field theory a “quantum group” arises from the 2-Hilbert space. In [F1] we carry out detailed computations for gauge theory with finite gauge group.

We end with some remarks about Chern-Simons theory with *continuous* gauge group. Then (2.2), (2.8), and (2.10) are replaced by integrals over the space of connections mod equivalence. Equation (2.2) is then the path integral heuristic given by Witten [W1]. Reshetikhin and Turaev [RT] subsequently gave an explicit computable formula in terms of quantum group data and proved that it is a topological invariant. Note that the formal integral (2.2) is over an infinite dimensional space. By contrast, the integral (2.8) reduces, after symplectic reduction, to an integral over the finite dimensional moduli space \mathcal{M}_Y of flat connections on Y . The integrand is a line bundle with connection whose curvature is the symplectic form on \mathcal{M}_Y . In this situation we may imagine that (2.8) is a formal expression for geometric quantization. The use of an integral is justified by the existence of gluing laws—Verlinde’s formula [V]. By now there are various proofs of at least special cases of these gluing laws.

What about (2.10)? If S is a circle then up to equivalence a connection is determined by its holonomy, which is a conjugacy class in the gauge group G . So (2.10) is an integral over a compact Lie group G with some invariance under the adjoint action. The integrand is a geometric object over G , “invariant” under the adjoint action, with a “connection” whose curvature is the canonical biinvariant 3-form on G constructed from the starting data $\lambda \in H^4(BG)$. (See [Bry] for one description.) I think it is reasonable to imagine that there is a geometric process—analogueous to geometric quantization—which constructs the 2-Hilbert space $\mathcal{E}(S)$ from this “gerbe bundle with connection”, but for now this remains a mystery.

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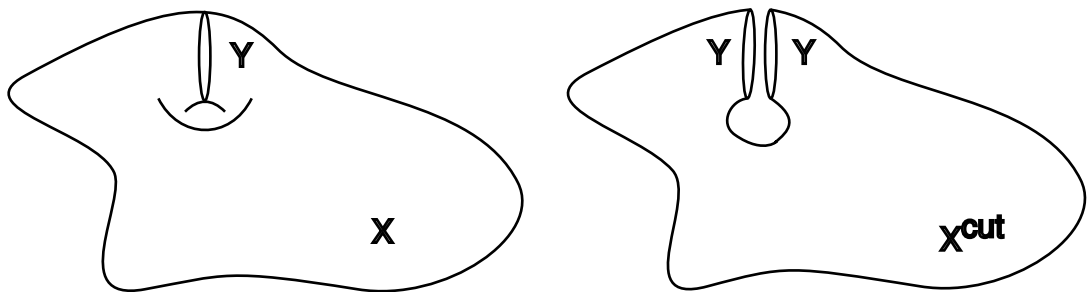


Figure 1: Cutting a manifold X along Y